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THE OPTIMAL DAMPING OF THE OSCILLATIONS OF ELASTIC BODIES†

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The problems of the optimal control (optimal, that is, in the sense of an integral quadratic functional) of the oscillations of an elastic body due to perturbations of a certain class are considered. Two types of problems are investigated: the limiting possibilities of control and the minimum of the guaranteed result. It is shown that the minimum of the guaranteed result has a lower limit given by the solution of the problem of the limiting possibilities of the control. Methods of solving these problems are proposed and examples of calculations for a specific elastic system are given.

Fairly effective methods of finding the optimal control of oscillations of elastic bodies have now been developed in the case when all the characteristics of the body and the external perturbation acting on it are completely known [1–4]. However, in many problems of the optimal damping of the oscillations of elastic structures complete information on the perturbation is not usually available. If only a set containing these perturbations is known, it is more convenient to formulate optimal problems using the principle of the minimum of the guaranteed result [5]. When analysing actual control systems it is important to estimate their limiting possibilities first. When the control is calculated using the principle of the minimum of a guaranteed result, the limiting possibilities of the control should be estimated taking into account the whole set of external perturbations acting on the controlled object.

1. FORMULATION OF THE PROBLEMS

Consider an elastic body [6] described by the initial-boundary-value problem

$$\rho(x)\frac{\partial^2 z}{\partial t^2}+B\frac{\partial z}{\partial t}+Cz=q^{\mathrm{T}}(x)u+r^{\mathrm{T}}(x)\upsilon(t) \tag{1.1}$$

$$z(x,t_0)=\dot z(x,t_0)=0 \tag{1.2}$$

$$H_\alpha z|_\Gamma=0,\quad \alpha=1,2,\ldots,a \tag{1.3}$$

Here $z(x, t)$ is the displacement of the elastic body from the state of equilibrium at the point $x\in\Omega$ when $t\in[t_0,\infty)$, Ω is the region, with fairly smooth boundary Γ, bounded in R^n, occupied by the body, B and C are linear differential operators of order $2a$, $\rho(x)$ is a continuous positive function characterizing the density of the elastic body, u and υ are m- and k-dimensional column vectors characterizing the controlling forces and the external perturbation, $q^{\mathrm{T}}(x)$, $r^{\mathrm{T}}(x)$ are m- and k-dimensional row vectors characterizing the strength of the control and of the perturbation, respectively, at the point x of the region Ω, and H_α are linear differential operators of order no higher than the order of the operator C.

We will assume that the stiffness operator C and the damping operator B are linearly related, i.e. $B = bC$, where b is a positive number. We will assume that the operator C is bounded, symmetric and positive-definite in the corresponding Sobolev-function space, satisfying boundary conditions (1.3). We will further assume that $\upsilon(t)$ is a piecewise-continuous vector function, each component of which is a function which is absolutely integrable in the interval $[0,\infty)$. The following conditions are also satisfied

$$\upsilon(t)\equiv 0\quad\text{for}\quad t<0;\quad \int_0^\infty \upsilon^{\mathrm{T}}P_0\upsilon\,dt\leqslant S_0^2 \tag{1.4}$$

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Here P_0 is a positive-definite symmetric (k,k)-matrix. The functions $\upsilon(t)$, which specify the external perturbations and which satisfy the above conditions, will belong to the class S. As regards the vector function $u = u(t)$ we will assume that it is piecewise-continuous and each of its components is absolutely integrable in the interval $(-\infty, \infty)$. We will also assume that an "ideal control", capable of leading the external perturbation, can be engaged at any instant of time $t_0 < 0$ preceding the instant when the perturbation starts, and when $t < t_0$ the identity $u(t) \equiv 0$ holds. The functions $u(t)$ which define the control and which satisfy these conditions will belong to the class D.

We will assume that each component of the vector function $q(x)$ is a Dirac delta-function $\delta(x - x_p)$, $x_p \in \Omega$. This means physically that the control is concentrated at a specified finite number of points of the elastic body. The vector function $r(x)$ specifies the distribution of the external perturbations acting on the elastic body. The function $r(x)$ is assumed to be bounded and integrable in Ω.

The functional characterizing the vibration activity of the elastic body and defined on the direct product $D \times S$ will be taken in the form

$$W[u(\cdot),\upsilon(\cdot)] = \int_{-\infty}^{\infty} w(t)dt \tag{1.5}$$

$$w(t) = \int_{\Omega}\left\{z(x,t)Cz(x,t) + \rho(x)\left[\frac{\partial z(x,t)}{\partial t}\right]^2\right\}dx$$

where the function $w(t)$, apart from a constant factor, is identical with the sum of the kinetic and potential energies of the oscillations of the elastic body.

We will now formulate the problem of the optimal damping of the oscillations of the elastic body. Problem 1 (the problem of the limiting possibilities of the control) consists of determining the quantity

$$W^0 = \sup_{\upsilon(\cdot)\in S}\inf_{u(\cdot)\in D} W[u(\cdot),\ \upsilon(\cdot)] \tag{1.6}$$

Hence to solve problem 1 we must first determine, for each perturbation from class S, the optimal control which minimizes the functional (1.5), and then maximize it with respect to all the actions $\upsilon(t)$.

To formulate problem 2 (the problem of optimizing the guaranteed quality) we must, in addition, determine the class of the control D_1. Unlike class D, class D_1 will only include those controls which can be realized in principle. We will assume that first, the control begins to act simultaneously with the external perturbation at the instant $t_0 = 0$, and second, the control u can be determined at any instant of time $t^* > 0$, if information is available only on the action $\upsilon(t)$ and the state of the elastic body in the time interval $[0, t^*]$. In particular, the relation between the control u, the perturbation $\upsilon(t)$ and the state of the elastic body $\{z(x, t), \dot{z}(x, t)\}$ is expressed by a functional relationship.

For any action $\upsilon(t)$ from the class S, by solving the initial-boundary-value problem (1.1)–(1.3) with a control from class D_1, we can express the control u as a function of time t. In addition, confining ourselves to class D_1, we will assume that the functions $u(t)$ obtained in this way are piecewise-continuous vector functions with components that are absolutely integrable in the interval $[0, \infty)$.

We will now formulate problem 2: it is required to obtain a control $u^0(\cdot) \in D_1$ such that

$$\sup_{\upsilon(\cdot)\in S} W[u^0(\cdot),\ \upsilon(\cdot)] = \inf_{u(\cdot)\in D_1}\sup_{\upsilon(\cdot)\in S} W[u(\cdot),\upsilon(\cdot)] \tag{1.7}$$

We will investigate later what relation exists between problems 1 and 2. We have

$$\sup_{\upsilon(\cdot)\in S}\inf_{u(\cdot)\in D} W[u(\cdot),\upsilon(\cdot)] \leqslant \sup_{\upsilon(\cdot)\in S}\inf_{u(\cdot)\in D_1} W[u(\cdot),\upsilon(\cdot)]$$

Using the inequality which is well known in the theory of antagonistic games [7], relating the maximin and the minimax, we obtain

$$\sup_{\upsilon(\cdot)\in S}\inf_{u(\cdot)\in D_1} W[u(\cdot),\upsilon(\cdot)] \leqslant \inf_{u(\cdot)\in D_1}\sup_{\upsilon(\cdot)\in S} W[u(\cdot),\upsilon(\cdot)]$$

Putting

$$\inf_{u(\cdot)\in D_1}\sup_{\upsilon(\cdot)\in S} W[u(\cdot),\upsilon(\cdot)] = W_*^0$$

we finally obtain $W^0 \leqslant W_*^0$. In other words, the value of the guaranteed result W_*^0 cannot be smaller than the value of W^0, which gives a solution of the problem of the limiting possibilities of the control.

2. PROBLEM 1

We will seek a solution of the initial-boundary-value problem (1.1)–(1.3) in the form of a series of eigenfunctions of the corresponding boundary-value problem

$$Cf(x) = \lambda\rho(x)f(x), \quad (H_\alpha f)|_\Gamma = 0, \quad \alpha = 1,2,\ldots,a$$

We know [8], that this problem has a denumerable number of eigenvalues λ_μ, to which there corresponds a family of eigenfunctions $f_\mu(x)$, forming a complete system in the corresponding Sobolev space. Note also that, by virtue of the fact that the operator C is positive-definite, for the eigenfunctions $f_\mu(x)$ and $f_\nu(x)$, corresponding to the different eigenvalues, we have

$$\int_\Omega \rho(x)f_\mu(x)f_\nu(x)dx = 0$$

By appropriate normalization of the eigenfunctions we can obtain that

$$\int_\Omega \rho(x)f_\mu^2(x)dx = 1$$

Thus, the solution of problem (1.1)–(1.3) can be represented in the form of a series

$$z(x,t) = \sum_{\mu=1}^{\infty} T_\mu(t)f_\mu(x)$$

Substituting the series into (1.1), multiplying both sides of the equation successively by $f_\mu(x)$ ($\mu = 1, 2, \ldots$) and integrating it over the whole region Ω, we obtain a denumerable system of ordinary differential equations

$$\ddot{T}_\mu + b\lambda_\mu\dot{T}_\mu + \lambda_\mu T_\mu = Q_\mu^{\mathrm{T}}u(t) + R_\mu^{\mathrm{T}}\upsilon(t), \quad \mu \geqslant 1 \tag{2.1}$$

with initial conditions $T_\mu(t_0) = \dot{T}_\mu(t_0) = 0$, where the row vectors Q_μ^{T} and R_μ^{T} are given by

$$Q_\mu^{\mathrm{T}} = \int_\Omega f_\mu(x)q^{\mathrm{T}}(x)dx, \quad R_\mu^{\mathrm{T}} = \int_\Omega f_\mu(x)r^{\mathrm{T}}(x)dx, \quad \mu \geqslant 1$$

The function $w(t)$ in (1.5) takes the form

$$w(t) = \sum_{\mu=1}^{\infty} [\dot{T}_\mu^2 + \lambda_\mu T_\mu^2]$$

Further, we will use the method described in [9] to solve problem. We multiply each of the equations (2.1) by $\exp(-i\omega t)$ and integrate the relations obtained with respect to the variable in the infinite interval $(-\infty, \infty)$. In terms of Fourier transforms we obtain

$$(-\omega^2 + ib\omega\lambda_\mu + \lambda_\mu)\theta_\mu(\omega) = Q_\mu^{\mathrm{T}}Y(\omega) + R_\mu^{\mathrm{T}}V(\omega), \quad \mu \geqslant 1 \tag{2.2}$$

Here $\theta_\mu(\omega)$, $Y(\omega)$, $V(\omega)$ are the Fourier transforms of the functions $T_\mu(t)$ and the vector functions $u(t)$ and $\upsilon(t)$. For the further analysis it is convenient to write system (2.2) in the matrix form

$$\Lambda\Theta(\omega) = QY(\omega) + RV(\omega) \tag{2.3}$$

where Λ is an infinite-dimensional diagonal matrix with elements $(-\omega^2 + ib\omega\lambda_\mu + \lambda_\mu)$, $\Theta(\omega)$ is an infinite dimensional column vector with elements $\theta_\mu(\omega)$, and Q and R are matrices with an infinite number of rows, each of which is the row vector Q_μ^{T} and R_μ^{T}, respectively.

We will denote the columns of the matrix Q by ξ_i and the elements of the column vector $Y(\omega)$ by $y_j(\omega)$ (here $j = 1, 2, \ldots, m$). We will consider the case when, among all the m columns ξ_j, m_1 of them are linearly independent, while the remaining $m - m_1$ are linear combinations of the form

$$\xi_j = \sum_{l=1}^{m_1} C_{jl}\xi_l, \quad m_1 \leqslant j \leqslant m \tag{2.4}$$

Without loss of generality we will assume that the first m_1 columns of ξ_j are linearly independent. Then

$$QY(\omega) = \sum_{j=1}^{m} \xi_j y_j(\omega) = \sum_{j=1}^{m_1} \xi_j y_j(\omega) + \sum_{j=m_1+1}^{m} \xi_j y_j(\omega)$$

Using (2.4) we finally obtain

$$QY(\omega) = \sum_{j=1}^{m_1} \xi_j \left[y_j(\omega) + \sum_{l=m_1+1}^{m} C_{lj} y_l(\omega) \right] = Q_0 U(\omega)$$

where Q_0 is a matrix in which the columns are linearly independent and $U(\omega)$ is a column vector with components

$$\tilde{y}_j(\omega) = y_j(\omega) + \sum_{l=m_1+1}^{m} C_{lj} y_l(\omega), \quad j = 1, \ldots, m_1 \tag{2.5}$$

Note that if all the columns of the initial matrix Q are linearly independent, we have $Q_0 = Q$, $U(\omega) = Y(\omega)$.

Thus, system (2.3) takes the form

$$\Lambda\Theta(\omega) = Q_0 U(\omega) + RV(\omega) \tag{2.6}$$

We will not consider further the case when all the elements of the matrix Q are zero. This means that the initial system is essentially uncontrollable. Such a situation arises, for example, when the controls act at clamped points of the boundary Γ of the elastic body.

From system (2.6) we obtain

$$\Theta(\omega) = \Lambda^{-1}[Q_0 U(\omega) + RV(\omega)]$$

Using Parseval's equality [10] and taking the last relation into account we obtain

$$\int_{-\infty}^{\infty} w(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)d\omega$$

where

$$G(\omega) = U^{\mathrm{T}} F_1 U^* + V^* F_2 V + U^{\mathrm{T}} F_3 V^* + V^{\mathrm{T}} F_4 U^* \tag{2.7}$$

(the superscript T denotes the operation of transposition, and the asterisk indicates a complex conjugate quantity), and $F_1 = Q_0^{\mathrm{T}}\Lambda_0 Q_0$, $F_2 = R^{\mathrm{T}}\Lambda_0 R$, $F_3 = F_4^{\mathrm{T}} = Q_0\Lambda_0 R$, while the diagonal infinite-dimensional matrix Λ_0 has the elements $(\omega^2 + \lambda_\mu) \times [(\omega^2 - \lambda_\mu)^2 + (b\lambda_\mu\omega)^2]^{-1}$.

We will minimize the right-hand side of (2.7) with respect to U. We determine the partial derivatives of G with respect to U and U^*

$$\frac{\partial G}{\partial U} = F_1 U^* + F_2 V^*, \quad \frac{\partial G}{\partial U^*} = F_1^{\mathrm{T}} U + F_4^{\mathrm{T}} V$$

Equating to zero the relations obtained, we obtain

$$U^* = -F_1^{-1} F_3 V^*, \quad U = -(F_1^{\mathrm{T}})^{-1} F_4^{\mathrm{T}} V \tag{2.8}$$

We put $F_0 = -F_1^{-1}F_3 = -(F_1^{\mathrm{T}})^{-1}F_4^{\mathrm{T}}$. Note that expressions (2.8) only have meaning if the matrix F_1 is non-degenerate.

We will show that this is indeed so. We introduce an infinite-dimensional diagonal matrix Λ_1 such that $\Lambda_1\Lambda_1 = \Lambda_0$. We also define the matrix $Q_1 = \Lambda_1 Q_0$. Note that if the columns of the matrix Q_0 are linearly independent, the columns of the matrix Q_1 are also linearly independent. At the same time $F_1 = Q_1^{\mathrm{T}} Q_1$ is the Gram matrix [11], the determinant of which is non-zero by virtue of the fact that the columns of Q_1 are linearly independent. Consequently, the matrix F_1 is non-degenerate.

Hence, for any ω, expression (2.7) takes an extremal value if the function U is defined in accordance with (2.8). To investigate the nature of this extremum we obtain

$$\frac{\partial^2 G}{\partial U^2} = \frac{\partial^2 G}{(\partial U^*)^2} = 0, \quad \frac{\partial^2 G}{\partial U \partial U^*} = F_1$$

We will show that F_1 is a positive-definite matrix. In fact, when $V = 0$ by definition $G(\omega)$ is of Hermitian form $U^{\mathrm{T}} F_1 U^* \geqslant 0$. Consequently, all the principal minors of the matrix F_1 are non-negative [11]. On the other hand, by the Gram criterion [11] they are not equal to zero. Thus the principal minors of the matrix F_1 are positive. Hence, by Silvester's criterion it follows that F_1 is positive definite. Consequently, the extremum obtained is a minimum.

Substituting (2.8) into (2.7) we obtain

$$G(\omega) = V^{\mathrm{T}} G_0(\omega) V^*, \quad G_0(\omega) = F_2 - F_3^{\mathrm{T}} F_1^{-1} F_3$$

Analysis of the matrices G_0 and F_0 shows that their elements are continuous functions of ω, bounded in the interval $(-\infty, \infty)$. Hence the complex vector function $U(\omega)$ from relations (2.7) is the Fourier transform of a certain vector function $\tilde{u}_0(t)$, from which, using (2.5), we can determine the m-dimensional vector function $u_0(t)$, from which, using (2.5), we can determine the m-dimensional vector function $u_0(t)$, which specifies the control in (1.1).

To find $u_0(t)$ from the known $\tilde{u}_0(t)$ we can proceed, for example, as follows: the first m_1 components of $u_0(t)$ are put equal to the components $\tilde{u}_0(t)$, and the subsequent $m - m_1$ components of $u_0(t)$ are equated to zero.

Thus, we have the inequality

$$\inf_{u(\cdot) \in D} W[u(\cdot), \upsilon(\cdot)] \geqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} V^{\mathrm{T}} G_0(\omega) V^* d\omega$$

If $u_0(t)$ belongs to class D, the last inequality becomes an equality. However, $u_0(t)$ may also not belong to class D.

Indeed, the function

$$u_\varepsilon^{\sim}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_0(\omega) V(\omega) e^{i\omega t} d\omega$$

may, generally speaking, differ from identical zero when $t < t_0$ for any finite value of t_0. Consequently, the corresponding vector function $u_0(t)$ does not belong to D.

In this case, we construct a minimizing sequence of functions

$$u_\varepsilon^{\sim}(t) = \begin{cases} \tilde{u}_0(t), & t \geqslant t_0^\varepsilon \\ 0, & t < t_0^\varepsilon \end{cases}$$

where t_0^ε is defined by the equation

$$\int_{-\infty}^{t_0^\varepsilon} \| u_0^{\sim}(t) \|_1 \, dt = \varepsilon$$

($\| \cdot \|_1$ is the norm of the vector). We define the Fourier transform of the function $\tilde{u}_\varepsilon(t)$

$$U_\varepsilon(\omega) = U(\omega) + \Omega_\varepsilon(\omega) = F_0(\omega) V(\omega) + \Omega_\varepsilon(\omega)$$

$$\left(\Omega_\varepsilon(\omega) = -\int_{-\infty}^{t_0^\varepsilon} \tilde{u}_0(t) e^{-i\omega t} dt \right)$$

Hence it follows that $\| \Omega_\varepsilon(\omega) \|_1 \leqslant \varepsilon$. Substituting $U_\varepsilon(\omega)$ into (2.7) we obtain

$$G(\omega) = V^{\mathrm{T}}(\omega)G_0(\omega)V^*(\omega) + \Omega_\varepsilon^{\mathrm{T}}(\omega)F_1(\omega)\Omega_\varepsilon^*(\omega)$$

Thus

$$W[u_\varepsilon^-(\cdot), \upsilon(\cdot)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} V^{\mathrm{T}}(\omega)G_0(\omega)V^*(\omega)d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Omega_\varepsilon^{\mathrm{T}}(\omega)F_1(\omega)\Omega_\varepsilon^*(\omega)d\omega$$

Since

$$\int_{-\infty}^{\infty}\Omega_\varepsilon^{\mathrm{T}}(\omega)F_1(\omega)\Omega_\varepsilon^*(\omega)d\omega \leqslant \varepsilon^2\int_{-\infty}^{\infty}\| F_1(\omega)\|_1\, d\omega$$

while the integral on the right-hand side of this inequality, as a simple analysis shows, converges, allowing ε to approach zero, we obtain

$$\lim_{\varepsilon\to 0} W[u_\varepsilon^-(\cdot), \upsilon(\cdot)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} V^{\mathrm{T}}(\omega)G_0(\omega)V^*(\omega)d\omega = \inf_{u(\cdot)\in D} W[u(\cdot), \upsilon(\cdot)]$$

while the quantity required in problem 1 is

$$W^0 = \frac{1}{2\pi}\sup_{\upsilon(\cdot)\in S}\int_{-\infty}^{\infty} V^{\mathrm{T}}(\omega)G_0(\omega)V^*(\omega)d\omega$$

We will now maximize the last integral with respect to $\upsilon(\cdot) \in S$.
Using inequality (1.4) and Parseval's equality we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} V^{\mathrm{T}}(\omega)P_0V^*(\omega)d\omega \leqslant S_0^2$$

Consider the two Hermitian forms $V^{\mathrm{T}}(\omega)G_0(\omega)V^*(\omega)$ and $V^{\mathrm{T}}(\omega)P_0V^*(\omega)$. Since the matrix P_0 is positive definite, a matrix $\Phi(\omega)$ exists [11] which specifies the transformation of the Hermitian form G_0 to the sum of squares, and P_0 to canonical form. Note that the matrix $\Phi(\omega)$ has real elements, since the matrix $G_0(\omega)$ has real elements. Thus we have

$$V(\omega) = \Phi(\omega)\Xi(\omega), \quad V^{\mathrm{T}}(\omega)G_0(\omega)V^*(\omega) = \Xi^{\mathrm{T}}(\omega)\Psi(\omega)\Xi^*(\omega)$$

$$V^{\mathrm{T}}(\omega)P_0V^*(\omega) = \Xi^{\mathrm{T}}(\omega)\Xi^*(\omega)$$

Here $\Psi(\omega)$ is a diagonal matrix with elements $\Psi_i(\omega)$ ($j = 1, 2, \ldots, k$).
Suppose that when $j = l$ and $\omega = \omega_0$ a maximum value is reached

$$\Psi^+ = \max_{j=[1,k]}\ \max_{\omega\in(-\infty,\infty)} \Psi_j(\omega)$$

It can then be shown [9] that the required solution of problem 1 is

$$W^0 = S_0^2\Psi^+$$

To conclude this section we will briefly formulate the main stages of the solution of the problem of the limiting possibilities of the control.

1. Obtain the eigenvalues λ_μ and the eigenfunctions $f_\mu(x)$ of the boundary-value problem;
2. set up the matrices Q and R;
3. discarding the linearly dependent columns in the matrix Q, form the matrix Q_0;
4. obtain the matrix $G_0(\omega)$;
5. obtain the transformation $\Phi(\omega)$ which reduces the matrix $G_0(\omega)$ to diagonal form;
6. determine Ψ^+.

When solving the problem numerically we must, of course, for a given accuracy, confine ourselves to a finite set of eigenfunctions of the boundary-value problem.

3. PROBLEM 2

It was not possible to solve problem 2 in general form in the class of controls D_1. We will confine ourselves to some special cases. Consider the parametric family of linear controls of the form

$$u_j = -\gamma_j \dot{z}(x_j,t) - \sigma_j z(x_j,t) \quad (j=1,2,\dots,m) \tag{3.1}$$

where γ_j, σ_j are non-negative parameters and x_j is the point on the elastic body at which the control u_j acts. After changing from the boundary-value problem to equations of the type (2.1) we obtain

$$T_\mu^{\cdot\cdot} + b\lambda_\mu T_\mu^{\cdot} + \lambda_\mu T_\mu = -\sum_{\nu=1}^{\infty}[T_\nu^{\cdot} \sum_{j=1}^{m} Q_{\mu j} Q_{\nu j}\gamma_j + T_\nu \sum_{j=1}^{m} Q_{\mu j} Q_{\nu j}\sigma_j] + R_\mu^{\mathrm{T}} \upsilon(t) \tag{3.2}$$

where $Q_{\mu j} = f_\mu(x_j)$.

The problem takes the following form: it is required to find parameters γ_j^0 and σ_j^0 such that

$$\sup_{\upsilon(\cdot)\in S} W[\gamma^0, \sigma^0, \upsilon(\cdot)] = \inf_{\gamma,\sigma\geqslant 0} \sup_{\upsilon(\cdot)\in S} W[\gamma,\sigma,\upsilon(\cdot)]$$

Here γ and σ are m-dimensional vectors with components $\{\gamma_j\}$ and $\{\sigma_j\}$.

To construct an approximate solution we will confine ourselves to a finite number N of equations in system (3.2). Below it will be more convenient to write this system in vector form

$$T^{\cdot\cdot} + B^N T^{\cdot} + C^N T = R^N \upsilon(t) \tag{3.3}$$

where T is a column vector with components $\{T_1, \dots, T_N\}$ and B^N and C^N are $(N\times N)$ matrix, each row of which $b_{\mu\nu}^N$ and $c_{\mu\nu}^N$ are defined as follows:

$$b_{\mu\nu}^N = b\lambda_\nu \delta_{\mu\nu} + \sum_{j=1}^{m} Q_{\mu j} Q_{\nu j}\gamma_j, \quad c_{\mu\nu}^N = \lambda_\mu \delta_{\mu\nu} + \sum_{j=1}^{m} Q_{\mu j} Q_{\nu j}\sigma_j$$

where $\delta_{\mu\nu}$ is the Kronecker delta and R^N is an $(N\times k)$ matrix, each row of which is equal to

$$R_\mu^{\mathrm{T}} = \int_\Omega f_\mu(x) r^{\mathrm{T}}(x)dx, \quad \mu = 1,2,\dots,N$$

Changing to Fourier transforms in (3.3) and repeating arguments similar to those above, we obtain

$$W_S[\gamma,\sigma] = \sup_{\upsilon(\cdot)\in S} W[\gamma,\sigma,\upsilon(\cdot)] = S_0^2 \max_{\omega\in(-\infty,\infty)} \max_{l\in(1,k)} \Psi_l^0(\omega)$$

where $\Psi_{l(\omega)}^0$ are the diagonal elements of the diagonal matrix $\Psi_{(\omega)}^0$, obtained from the matrix $A(\omega)$ by simultaneously reducing the Hermitian form $V^{\mathrm{T}}A(\omega)V^*$ and $V^{\mathrm{T}}P_0V^*$ to the sum of squares and canonical form. Here

$$A(\omega) = (R^N)^{\mathrm{T}} \Gamma_0^{\mathrm{T}}(\omega) L_0(\omega) \Gamma_0^*(\omega) R^N$$

$$\Gamma_0^{-1}(\omega) = (-\omega^2 E + i\omega B^N + C^N), \quad L_0(\omega) = \omega^2 E + L_1$$

(E is the identity matrix and L_1 is a diagonal matrix with elements λ_μ, $\mu = 1, 2, \dots, N$).

Hence, we have constructed the function $W_S[\gamma, \sigma]$ of the parameters γ, σ. To solve problem 2 we must minimize this function with respect to γ, σ. Note that the function $W_S[\gamma, \sigma]$ is not a continuous function, and hence it is best to use methods of non-differentiable optimization [12] to minimize it. We put

$$W_S^0 = \inf_{\gamma,\,\sigma\geqslant 0} W_S[\gamma,\sigma]$$

and introduce the relation

$$\alpha_0 = W_S^0 / W^0 \geqslant 1$$

which shows to what extent the indicator of guaranteed quality provided by a control of the form (3.1) is close to the maximum possible value W^0.

Another case is related to a control of the form

$$u = \chi \upsilon \tag{3.4}$$

where χ is a constant $(m \times k)$-matrix. After changing from the boundary-value problem to equations of the form (2.1) we obtain

$$T_\mu^{\cdot\cdot} + b\lambda_\mu T_\mu^{\cdot} + \lambda_\mu T_\mu = (Q_\mu^{\mathrm{T}} \chi + R_\mu^{\mathrm{T}}) \upsilon(t) \tag{3.5}$$

Problem 2 takes the following form: it is required to find elements of the matrix χ^0 such that

$$\sup_{\upsilon(\cdot) \in S} W[\chi^0, \upsilon(\cdot)] = \inf_\chi \sup_{\upsilon(\cdot) \in S} W[\chi, \upsilon(\cdot)]$$

To construct an approximate solution, as in the previous case, we will confine ourselves to a finite number N of equations in system (3.5). Changing in the truncated system to Fourier transforms and repeating arguments similar to those above, we obtain

$$W_+[\chi] = S_0^2 \sup_{\upsilon(\cdot) \in S} W[\chi, \upsilon(\cdot)] = S_0^2 \max_{\omega \in (-\infty, \infty)} \max_{l \in (1, k)} \Psi_l^+(\omega)$$

where $\Psi_l^+(\omega)$ are the diagonal elements of the matrix $\Psi^+(\omega)$, obtained from the matrix $A_+(\omega)$ by simultaneously reducing the Hermitian forms $V^{\mathrm{T}} A_+(\omega) V^*$ and $V^{\mathrm{T}} P_0 V^*$ to the sum of squares and the canonical form. Then

$$A_+(\omega) = (R_+^N)^{\mathrm{T}} \Gamma_+^{\mathrm{T}}(\omega) L_0(\omega) \Gamma_+^*(\omega) R_+^N,$$

where $\Gamma_+^{-1}(\omega)$ is a diagonal matrix with elements equal to $-\omega^2 + i\omega b\lambda_\mu + \lambda_\mu$ ($\mu = 1, 2, \ldots, N$) and R_+^N is an $N \times k$ matrix, each row of which is a row vector $Q_{\mu\chi}^{\mathrm{T}} + R_\mu^{\mathrm{T}}$, ($\mu = 1, 2, \ldots, N$). Thus, we have constructed the function $W_+[\chi]$. To solve problem 2 we need to minimize this function with respect to the elements of the matrix χ. We put

$$W_+^0 = \inf_\chi W_+[\chi]$$

and introduce the relation

$$\alpha_+ = W_+^0 / W^0 \geqslant 1$$

which has the same meaning as α_0.

Note that the two types of controls (3.1) and (3.4) considered correspond to two basically different approaches to the problem of reducing the vibrational activity of the bodies, known in the theory of vibration-protected systems as passive and active vibration isolation. Other methods of specifying the control are obviously also possible. We will not give any further examples, but will merely note that all possible forms of control forces can be divided into linear and non-linear ones depending on how the control depends on the deformation, the rate of deformation of the elastic body and the external perturbation. In the case of linear controls one can use the scheme for solving problem 2 described above. In the case of non-linear controls it is not possible to solve problem 2 in the initial formulation. However, one can proceed as follows: Instead of the initial class of forces S we choose a finite set S_q of external perturbations, satisfying conditions (1.4). This method is often encountered in practical engineering when non-linear systems are being investigated. Perturbations from the finite set S_q are usually called reference perturbations.

Thus, if a parametric family of non-linear controls is specified, the method of solving problem 2 consists of determining the functional in the set S_q numerically and subsequently minimizing the function of the parameters defining the control, obtained in this way. We emphasize that after reducing the partial differential equation to a finite-dimensional system we will have a non-linear system of ordinary differential equations. Hence, the functional must be calculated by numerical integration of this system. This optimization problem becomes extremely time consuming in computational respects when there is a fairly large number of reference perturbations. However, it must be borne in mind that in this case one can easily find the maximum possible value of the optimum.

Indeed, by determining, for any reference perturbation from S_q, its Fourier transform and using the approach described above one can solve the problem of the limiting possibilities of the control in the class of perturbations S_q.

Of course, the existence of such an estimate does not remove the computational difficulties involved in solving the optimization problem. Nevertheless, the estimate serves as a definite reference point and enables the process of optimization to be organized more effectively.

Note that the quantities W^0, W_S^0, W_+^0, obtained by solving problems 1 and 2, depend on the points x_j ($j = 1, 2, \ldots, m$) of the elastic body at which the control acts. Hence, in addition to the problems considered above we can formulate the problem of the optimum distribution of the controlling devices. Combined, for example, with problem 1, it will have the following form: it is required to obtain x_j^0 such that

$$W^0(x_j^0) = \min_{x_j \in \Omega} W^0(x_j^0)$$

4. EXAMPLE

Consider a uniform elastic string with internal damping. One end of the string is fixed to a certain solid (henceforth called the base), which moves rectilinearly as given by a certain law. The other end of the string is free. In addition, one of the points of the string, with coordinate x_0, is connected to the base through a controller. We will investigate the transverse oscillations of the string. In this case the differential operator $C = -c\partial^2/\partial x^2$, while the equations of transverse oscillations of the string have the form

$$\rho\frac{\partial^2 Z}{\partial t^2} - b\frac{\partial^2 Z}{\partial x^2 \partial t} - c\frac{\partial^2 Z}{\partial x^2} = q(x)u + \rho\upsilon$$

Here $Z(x, t)$ is the transverse displacement of a point of the string with coordinate x at the instant of time t, ρ is the density of the string, the positive parameters b and c represent the internal damping and elasticity of the string, $q(x) = \delta(x - x_0)$ ($x_0 \in (0, L]$, L is the length of the string), and $\upsilon = \upsilon(t)$ is the external perturbation, which is identical with the acceleration of the base, apart from sign. The boundary conditions have the form $Z(0, t) = Z_x'(L, t) = 0$. The matrix P_0 in (1.4) has a single element, equal to unity.

By replacing the variables

$$Z = L\tilde{Z}, \quad x = L\tilde{x}, \quad t = (\rho/c)^{1/2} L\tilde{t}$$

and subsequently dropping the tilde superscript, we can reduce the equations of motion to the form

$$\frac{\partial^2 Z}{\partial t^2} = \beta\frac{\partial^2 Z}{\partial x^2 \partial t} + \frac{\partial^2 Z}{\partial x^2} + q(x)u + \upsilon(t), \quad Z(0,t) = Z_x'(1,t) = 0.$$

Inequality (1.4) can be written as follows:

$$\int_0^\infty \upsilon^2(t)dt \leq \eta_0^2, \quad \beta = b(\rho c)^{-1/2} L^{-1}, \quad \eta_0^2 = S_0^2 L\left(\frac{\rho}{c}\right)^{3/2}$$

The integrand $w(t)$ from (1.5) will have the form

$$w(t) = \int_0^1 [(Z^{\cdot}(x,t))^2 + (Z_x'(x,t))^2]dx$$

while the functional

$$W[u(\cdot), \upsilon(\cdot)] = \kappa_0 \int_{-\infty}^{\infty} w(t)dt, \quad \kappa_0 = L^2(\rho c)^{1/2}$$

We will present the results of a numerical solution of problem 1. Without loss of generality we will put $\eta_0^2\kappa_0 = 1$.

Assuming that $\beta = 0.1$, the values of W^0 will depend on the parameter x_0 as follows:

x_0	0	0.01	0.1	0.2	0.4	0.6	0.8	1.0
W_0	26.6	9.23	1.12	0.475	0.123	0.0466	0.0975	0.146

The value $x_0 = 0$ corresponds to no control of the system.

As follows from these results, the minimum value of W^0 is reached when $x_0 \in [0.6, 0.8]$. By refining the value of the minimum we obtain $x_0^0 = 0.61$ and $W^0(x_0^0) = 0.0444$.

We will obtain a numerical solution of problem 2 when the linear control (3.1) is used. The solution of the problem of two-parameter optimization with $x_0 = 0.61$ gives the following results the optimum values of the parameters are $\gamma^0 = 3.85$ and $\sigma^0 = 0.878$, and the corresponding optimum value of the index of guaranteed quality and the ratio introduced above are

$$W_S^0 = W_S[\gamma^0, \sigma^0] = 0.163, \quad \alpha_0 = W_S^0 / W^0 \doteq 3.67$$

The use of a control of the form (3.4) in problem 2 with $x_0 = 0.61$ gives the following result: $\chi^0 = -0.76$, $W_+^0 = 0.0565$, and the corresponding value of the ratio is

$$\alpha_+ = W_+^0 / W^0 = 1.27$$

Hence, the control (3.4) in this case is preferable to the control (3.1).

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